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COAP 2018 Best Paper Prize

Each year, the editorial board of Computational Optimization and Applications selects a paper from the preceding year's publications for the Best Paper Award. In 2018, 93 papers were published by the journal. The recipients of the 2018 Best Paper Award are Christoph Buchheim (Technische Universität Dortmund), Renke Kuhlmann (University of Wisconsin, Madison), and Christian Meyer (Technische Universität Dortmund) for their paper "Combinatorial optimal control of semilinear elliptic PDEs" published in volume 70, pages 641–675. This article highlights the research related to the award winning paper.

In [1], the authors study the optimal control of semilinear partial differential equations (PDEs) over combinatorial constraints. This problem class models static diffusion processes that are controllable by switching heat sources on/off and where the switching itself may be constrained, e.g., by a knapsack condition. Solving this kind of problem to global optimality is extremely difficult, since the problem formulation contains both combinatorial and (infinitely many) nonlinear constraints. A straightforward approach would be to discretize the PDE over the domain. This, however, yields a large scale mixed-integer nonlinear (typically non-convex) optimization problem that can easily become intractable. Most approaches in the literature therefore are heuristic in nature; some are based on a linearization of the PDE [2,3]. For discrete controls changing over time, the Sum-Up Rounding approach has been proposed [4,5]. However, the latter is not able to deal with non-trivial combinatorial constraints allowing an exponential number of feasible switchings.

In their work, the authors of [1] focus on the PDE

$$Ay + g(y) = \sum_{i=1}^{n} u_i \psi_i \quad \text{in } \Omega \tag{1}$$

with Dirichlet and/or Neumann boundary conditions and a control $u \in \mathbb{R}^n$. Herein Ω denotes a bounded domain in $\mathbb{R}^d$, $d \in \mathbb{N}$, and A is a linear, elliptic operator. In addition, ψ_i are given form functions. Based on the main assumption of the paper that the operator g is pointwise non-decreasing and convex, the authors develop algorithms for globally solving the following three combinatorial optimal control problems.

The first part of the paper studies

$$\begin{aligned} \min \quad & c^\top u \\ \text{subject to} \quad & y(x) \geq y_{\min}(x) \quad \text{a.e. in } \Omega \\ & y \text{ solves (1) for } u \\ & u \in \mathcal{U} \end{aligned} \tag{2}$$

with a minimum state $y_{\min} \in L^1(\Omega)$ (e.g. minimum temperature) and $\mathcal{U} \subseteq \mathbb{Z}^n$ being any bounded combinatorial set. The authors show that the solution operator of (1) is pointwise concave in the control u which enables to design an outer approximation approach for (2). In each iteration valid cutting planes (based on tangents) are computed by an efficiently solvable linear PDE. Since concavity holds pointwise, usually infinitely many such cutting planes exist, which required further algorithmic considerations concerning the selection of appropriate cutting planes.

In the second part, the authors develop valid cutting planes for upper bounds $y_{\max} \in L^1(\Omega)$, i.e., for the optimal control problem

$$\begin{aligned} \min \quad & c^\top u \\ \text{subject to} \quad & y(x) \leq y_{\max}(x) \quad \text{a.e. in } \Omega \\ & y \text{ solves (1) for } u \\ & u \in \mathcal{U}. \end{aligned} \tag{3}$$

Under the further assumption of binary controls, i.e. $\mathcal{U} \subseteq \{0, 1\}^n$, and non-negative form functions, the authors show that the solution operator of (1) is also submodular in the control u. Hence, each nonlinear constraint on the control variables u derived at a point x is essentially equivalent to a finite (but exponential) number of linear constraints, where the most violated one can be calculated efficiently, which can again be embedded into an outer approximation algorithm. To obtain this result, the authors had to combine advanced methods from both optimal control and discrete optimization.

Finally, exploiting both concavity and submodularity, the authors can address any linear constraint in both controls and states as well as L^p-tracking type objective functions for any $p \in [1, \infty]$, i.e.,

$$\begin{aligned} \min \quad & \|y - y_d\|_{L^p(\Omega)} \\ \text{subject to} \quad & y \text{ solves (1) for } u \\ & u \in \mathcal{U} \end{aligned} \tag{4}$$

with a desired state $y_d \in L^p(\Omega)$, still for $\mathcal{U} \subseteq \{0, 1\}^n$. In all cases, the obtained algorithm for solving the problem to global optimality is finite.

The results published in [1] are stated for elliptic PDEs. However, they can be easily adapted to the case of a parabolic PDE of the form

$$\partial_t y + Ay + g(y) = \sum_{i=1}^{n} u_i \psi_i \quad \text{in } \Omega$$

as long as the controls $u \in \mathcal{U}$ are constant over the entire time horizon. An interesting topic for future work is how far it is possible to deal with combinatorial controls changing over time. Another important subject of future research is to weaken the rather restrictive convexity assumption on the nonlinear function g in order to cover further real-world application problems.

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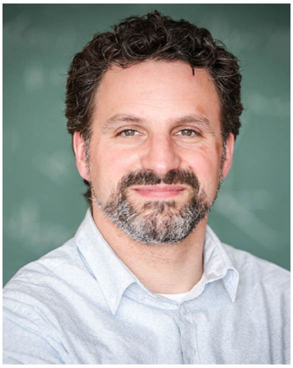
Christoph Buchheim received his Ph.D. in Computer Science from the University of Cologne in 2003. Afterwards, he was a postdoctoral researcher at IASI-CNR in Rome and visiting researcher at the University of Bologna, the University of Bonn and the Technical University of Munich. Since 2009, he is Full Professor at the Faculty for Mathematics of TU Dortmund and chair of the Discrete Optimization group. His research focus is on optimization problems combining nonlinear and discrete structures.

Renke Kuhlmann earned a Ph.D. in 2018 from the University of Bremen, where he developed the interior-point method of the nonlinear programming solver WORHP. He further had a research position at TU Dortmund and is currently working as a postdoctoral associate at the University of Wisconsin–Madison.



Christian Meyer is professor for continuous optimization at TU Dortmund. He studied mechanical and physics engineering at TU Berlin, where he also received his PhD in mathematics in 2006. From 2006 to 2009, he was postdoctoral researcher at the Weierstrass Institute in Berlin. Before he moved to Dortmund, he was assistant professor at TU Darmstadt from 2009 to 2011. His research interests include optimization and optimal control of partial differential equations and variational inequalities, with a particular emphasis on problems arising in computational mechanics.