Rational Expectations with Endogenous Information

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Abstract

This paper presents a general solution method for rational expectations models with dispersed information when the information process is endogenous. First, I show how to solve models with exogenous information by applying a single matrix equation. Next, I present an algorithm, *Signal Operator Iteration*, which solves the model when information is endogenous. I characterize conditions under which the solution is unique and the algorithm converges. Finally, I apply the solution method to a model of local information. Firms observe prices and quantities in their own market, but not the aggregate state of the economy. They must make inferences about aggregate shocks through the impacts on endogenous prices. Observed prices do not fully reveal fundamental shocks, so money is non-neutral. The monetary authority's policy rule determines the size of the real effects of nominal shocks, by affecting how informative prices are about the aggregate state. All noisy signals are driven by fundamental shocks - observable ex-post to the econometrician - so data can discipline the information structure. The model is calibrated using US industry-level panel data.

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1 Introduction

Dynamic macroeconomic models with dispersed information have recently generated significant research interest for their ability to simultaneously explain macroeconomic patterns such as inertia and volatility, while remaining useful for prediction by featuring micro-foundations and unique equilibria. To become more realistic, economists have started to incorporate endogenous information processes into these models, such as requiring agents to make inferences from equilibrium prices. But when the information process is endogenous, these models are typically difficult to solve, and have no general solution or even a guarantee of a unique equilibrium, particularly when models feature endogenous state variables. In this paper, I resolve that difficulty.

The paper's main contribution is methodological. I begin by deriving a general solution to a class of macroeconomic models with exogenous dispersed information. Then, I show that this solution can be applied iteratively when information is endogenous. I label this algorithm *Signal Operator Iteration*, and derive conditions under which it will converge to the unique equilibrium of the model. The algorithm is fast and easy to apply: it simply consists of repeated matrix operations. It can be applied to solve a general class of macroeconomic models; previous solution methods either required specialized settings, or approximations to the endogenous information process.

I apply the solution method to a model of *local information*: firms can observe prices and quantities in their local market, but not the aggregate state of the economy. Rather, they must make inference about aggregate shocks through their effects on local variables. Signals such as prices are endogenous, so Signal Operator Iteration must be used to solve the model. In equilibrium, money is non-neutral, and the size of its effects on real variables depends on the strength of the feedback from endogenous variables to signals.

Endogenous information is a powerful modeling tool because it can be parsimonious, and straightforward to take to the data. Models with exogenous processes for dispersed information require new parameters to characterize the time series properties of noisy signals. But when information is endogenous, the structure of the model governs the time series of the noisy signals.[1] And the shocks that confound inference of the aggregate state have a clear mapping in the data. For example, in this paper's local information model, idiosyncratic sectoral demand shocks obfuscate inference of aggregate demand. These demand shocks are measurable with sectoral price and quantity data from national accounts.

The literature has turned to several methods to solve dynamic models with endogenous information that highlight the importance of information exogeneity for macroeconomic dynamics, but these methods are limited to special cases.[2] Papers such as Kasa (2000), Acharya (2013), and Rondina and Walker (2015) examine a model in which they can use Blaschke root flipping to find the Wold representation of the information process. This method requires that there are the same number of shocks as signals; otherwise finding the Wold representation is more difficult. An alternative approach is to assume that shocks become common knowledge after some fixed number of periods, as in Hellwig and Venkateswaran (2009) or originally in Lucas (1972). Huo and

[1]In this context, the endogeneity of information refers to the endogenous determination fo the time series properties of the information observed by economic agents. This contrasts with the large literature of endogenous information acquisition, where agents choose to utilize a subset of available information. These theories have been applied to monetary questions in cases such as the the rational inattention literature followingSims (2003), or the sticky information literature following Mankiw and Reis (2002), Reis (2006), and Alvarez, Lippi, and Paciello (2011).

[2]A large literature has developed the solution methods for dispersed information models more generally, following the seminal work of Townsend (1983). Huo and Takayama (2016) review this literature.

Takayama (2016) use a finite ARMA to approximate the solution to an endogenous information problem, and show there is no finite ARMA representation to the true solution.

Nimark (2017) uses a similar iterative algorithm to calculate higher order expectations in a general asset pricing model with endogenous information. This algorithm is useful in asset pricing models characterized by a forward-looking Euler equation, but cannot be directly applied to macroeconomic models with backward-looking state variables.[3] Additionally, Nimark (2017) requires that agents either directly observe aggregate endogenous variables, or not at all. Signal Operator Iteration admits these options, but also allows agents to observe noisy signals of the aggregates. The drawback of this algorithm relative to Nimark (2017) is that some further conditions are necessary to guarantee convergence and uniqueness of equilibrium.

This paper join a large literature studying the relationship between information frictions and monetary non-neutrality. Lucas (1972) is the seminal paper and featured endogenous information in a static model. Research followed to translate the intuition to a dynamic setting, but Sargent (1991) suggests that the profession abandoned this approach due to the difficulty of solving these sorts of models. More recently, this question has been revisited by papers such as Woodford (2003), Angeletos and LaO (2009), Lorenzoni (2010), Makowiak and Wiederholt (2009), Angeletos, Iovino, and La'O (2016) and Melosi (2016) among many others that examine nominal rigidities in dispersed information frameworks. This paper also joins the larger literature on information frictions in macroeconomics, which Angeletos and Lian (2016) survey.

2 Exogenous Information

This section describes a linear method to solve macroeconomic models with exogenous information processes. This will be an intermediate step in the solution of models with endogenous information.

Consider a stationary linear macroeconomic model of the following form. The equilibrium conditions for agent i at time t are:

$$0 = E_{i,t}[B_{S0}X_{S,i,t} + B_{S1}X_{S,i,t+1} + B_{C0}X_{C,i,t} + B_{C1}X_{C,i,t+1} + B_{Z0}Z_{i,t} + B_{Z1}Z_{i,t+1}] \tag{1}$$

$X_{S,i,t}$ is an $n_S \times 1$ vector containing the endogenous state variables and $X_{C,i,t}$ is an $n_C \times 1$ vector containing the control variables at time t. The $m \times 1$ vector $Z_{i,t}$ contains variables that agent i takes as exogenous, including economic shocks and exogenous signals. The matrices $\{B_{S0}, B_{S1}, B_{C0}, B_{C1}, B_{Z0}, B_{Z1}\}$ contain coefficients corresponding to the $n \equiv n_C + n_S$ equilibrium conditions of the model. The expectations operator $E_{i,t}$ applies to the information set of a agent i at time t. Information is exogenous, so individual choices do not affect the information available to other agents.

A linear solution to the model is policy that expresses $X_{S,i,t}$ and $X_{C,i,t}$ as a function of exogenous variables Z_k for $k \leq t$ such that (1) holds with equality for all t. This is not necessarily a recursive policy function; it may depend on the entire history of Z_k. Specifically, define policy functions to be linear in the history of white noise innovations $w_{i,t}$ implied by the Wold decomposition of $Z_{i,t}$. Let $Z(L)$ be the Wold representation expressed as a polynomial in the lag operator L. Then $Z_{i,t}$ is given by

$$Z_{i,t} = Z(L)w_{i,t} \equiv \sum_{j=0}^{j=\infty} Z_j L^j w_{i,t} \tag{2}$$

[3] Nimark (2008) and Melosi (2016) use this algorithm to solve New Keynesian models with endogenous information but without endogenous state variables.

Stack the endogenous variables so that $\begin{pmatrix} X_{C,i,t} \\ X_{S,i,t} \end{pmatrix} \equiv X_{i,t}$, which is a $n \times 1$ vector. The policy function can be expressed as a polynomial in the lag operator:

$$X_{i,t} = X(L)w_{i,t} \equiv \sum_{j=0}^{j=\infty} X_j L^j w_{i,t} \tag{3}$$

Expressing policy functions in terms of information is convenient because forecasting is straightforward: $E_{i,t}[w_{i,t+k}] = 0$ for all $k > 0$.[4] Frequently the policy function is expressed in term of the history of signals, and this form is easily recovered because the Wold decomposition is invertible:

$$X_{i,t} = X(L)w_{i,t} = X(L)Z(L)^{-1}Z_{i,t} \tag{4}$$

where $\{X_j\}_{j=0}^{\infty}$ are $n \times m$ matrices.

When expressed in terms of innovations, the equilibrium condition (1) becomes

$$0 = [B_{X0}X(L)w_{i,t} + B_{X1}L^{-1}X(L)w_{i,t} + B_{Z0}Z(L)w_{i,t} + B_{Z1}L^{-1}Z(L)w_{i,t}]_+ \tag{5}$$

where $[\cdot]_+$ is the annihilation operator, which annihilates negative powers of L. The coefficient matrices are stacked so that $\begin{pmatrix} B_{C0} & B_{S0} \end{pmatrix} \equiv B_{X0}$ and $\begin{pmatrix} B_{C1} & B_{S1} \end{pmatrix} \equiv B_{X1}$.

The equilibrium policy function can be expressed as a linear function of the Wold decomposition. Before deriving the formula, some notation must be defined. Let Λ_C denote a diagonal matrix of the eigenvalues of $-B_{X1}^{-1}B_{X0}$ that are outside the complex unit circle, and let Λ_S denote a diagonal matrix of eigenvalues that are inside the complex unit circle. Let Q denote a matrix of eigenvectors of $-B_{X1}^{-1}B_{X0}$ ordered so that

$$-B_{X1}^{-1}B_{X0} = Q \begin{pmatrix} \Lambda_C & 0 \\ 0 & \Lambda_S \end{pmatrix} Q^{-1} \tag{6}$$

and let Q_C^{-1} denote the upper left $n_C \times n_C$ block of Q^{-1}. Define the matrices $\tilde{B}_k$ and $\tilde{Z}_k$ by

$$\tilde{B}_k \equiv \begin{cases} Q \begin{pmatrix} -\Lambda_C^k & 0 \\ 0 & 0 \end{pmatrix} Q^{-1} & k < 0 \\ Q \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_S^k \end{pmatrix} Q^{-1} & k \geq 0 \end{cases} \tag{7}$$

$$\tilde{Z}_k \equiv \begin{cases} -B_{X1}^{-1}(B_{Z1}Z_k + B_{Z0}Z_{k-1}) & k > 0 \\ -\sum_{k=1}^{\infty} \begin{pmatrix} Q_C\Lambda_C^{-k} & 0_{n_C \times n_S} \\ 0_{n_S \times n_C} & 0_{n_S \times n_S} \end{pmatrix} Q^{-1}\tilde{Z}_k & k = 0 \\ 0 & k < 0 \end{cases} \tag{8}$$

and lastly define the polynomials $\tilde{B}(L)$ and $\tilde{Z}(L)$ by

$$\tilde{B}(L) \equiv \sum_{j=-\infty}^{\infty} \tilde{B}_k L^k \qquad \tilde{Z}(L) \equiv \sum_{j=-\infty}^{\infty} \tilde{Z}_k L^k \tag{9}$$

[4] This is the Wiener-Kolmogorov prediction formula. See Hansen and Sargent (1981) for a description in the context of rational expectations models.

Theorem 1 *If B_{X1} is invertible and if $-B_{X1}^{-1}B_{X0}$ has n_C eigenvalues outside the unit circle, and n_S nonzero eigenvalues inside the unit circle, then the model has a unique solution and the policy function is given by $X(L) = [\tilde{B}(L)\tilde{Z}(L)]_+$.*

Proof. The equilibrium conditions (5) must hold for all realizations of the shocks, so it's possible to collect terms, restricting the values of the matrices $\{X_j\}_{j=0}^{\infty}$. This implies a recursive equation for $j \geq 1$:

$$0 = B_{X0}X_j + B_{X1}X_{j+1} + B_{Z0}Z_j + B_{Z1}Z_{j+1} \tag{10}$$

Left multiply by B_{X1}, substitute with $\tilde{Z}_j$, and rearrange to get

$$X_{j+1} = -B_{X1}^{-1}B_{X0}X_j + \tilde{Z}_{j+1} \tag{11}$$

for $j \geq 0$. Eigendecompose the matrix $-B_{X1}^{-1}B_{X0}$, ordering the eigenvalues as in (6), and left multiply by Q^{-1}:

$$Q^{-1}X_{j+1} = \begin{pmatrix} \Lambda_C & 0 \\ 0 & \Lambda_S \end{pmatrix} Q^{-1}X_j + Q^{-1}\tilde{Z}_{j+1}$$

The recursive relationship can now be separated into a stable recursive equation and an unstable recursive equation. Let $(Q^{-1}X)_{C,j}$ and $(Q^{-1}\tilde{Z})_{C,j}$ denote the first n_C rows of $Q^{-1}X_j$ and $Q^{-1}\tilde{Z}_j$ respectively. Then the unstable recursive equation is

$$(Q^{-1}X)_{C,j+1} = \Lambda_C(Q^{-1}X)_{C,j} + (Q^{-1}\tilde{Z})_{C,j+1} \tag{12}$$

And where $(Q^{-1}X)_{S,j}$ and $(Q^{-1}\tilde{Z})_{S,j}$ denote the corresponding last n_S rows, the stable recursive equation is

$$(Q^{-1}X)_{S,j+1} = \Lambda_S(Q^{-1}X)_{S,j} + (Q^{-1}\tilde{Z})_{S,j+1} \tag{13}$$

Because Λ_C is diagonal with all values outside the unit circle, the unstable recursive equation (12) allows $(Q^{-1}X)_{C,j}$ to be expressed as the infinite sum

$$(Q^{-1}X)_{C,j} = -\sum_{k=1}^{\infty} \Lambda_C^{-k}(Q^{-1}\tilde{Z})_{C,j+k} \quad \forall j \geq 0 \tag{14}$$

This is not an invertible equation for X_j, but at $j = 0$ there is a restriction that $X_{S,0} = 0_{n_S\times 1}$ because these dimensions correspond to the state variables. Thus the initial controls are exactly determined by

$$X_{C,0} = -Q_C \sum_{k=1}^{\infty} \Lambda_C^{-k}(Q^{-1}\tilde{Z})_{C,k} \tag{15}$$

Per equation (8), $X_0 = \tilde{Z}_0$.

Similarly, the stable recursive equation (13) implies the infinite sum

$$(Q^{-1}X)_{S,j} = \sum_{k=0}^{\infty} \Lambda_S^{k}(Q^{-1}\tilde{Z})_{S,j-k} \tag{16}$$

Stack equations (14) and (16) and redefine the indices to yield

$$\begin{pmatrix} (Q^{-1}X)_{C,j} \\ (Q^{-1}X)_{S,j} \end{pmatrix} = \sum_{k=-\infty}^{\infty} \begin{pmatrix} -\Lambda_C^{k}\mathbb{1}\{k<0\} & 0 \\ 0 & \Lambda_S^{k}\mathbb{1}\{k\geq 0\} \end{pmatrix} \begin{pmatrix} (Q^{-1}\tilde{Z})_{C,j-k} \\ (Q^{-1}\tilde{Z})_{S,j-k} \end{pmatrix}$$

Finally, left multiply by Q and substitute in $\tilde{B}_k$ to recover X_j:

$$X_j = \sum_{k=-\infty}^{\infty} \tilde{B}_k \tilde{Z}_{j-k} \quad \forall j \geq 0$$

which is equivalent to the convolution $X(L) = [\tilde{B}(L)\tilde{Z}(L)]_+$.

■

The requirement that $-B_{X1}^{-1}B_{X0}$ has n_C eigenvalues outside the unit circle is the Blanchard and Kahn (1980) condition: there must be as many unstable eigenvalues as there are contemporaneous jump variables for the equilibrium to be uniquely determined. This condition is also what prevents application of the Nimark (2017) solution algorithm when the model includes endogenous state variables. The Nimark condition requires that the matrix norm of $(-B_{X1}^{-1}B_{X0})^{-1}$ is less than one.[5] $(-B_{X1}^{-1}B_{X0})^{-1}$ has n_S eigenvalues outside the unit circle, and the norm of a matrix is weakly greater than the largest absolute eigenvalue, so if there are endogenous state variables then the Nimark algorithm cannot be directly applied.

3 Endogenous Information

This section describes how to solve macroeconomic models with endogenous information processes. A general solution algorithm is presented, and then conditions for it to be applied are characterized.

Suppose the signals $Z_{i,t}$ observed by agent i are a sum of exogenous signals $S_{Z,i,t}$ and endogenous signals $S_{X,i,t}$:

$$Z_{i,t} = S_{Z,i,t} + S_{X,i,t} \tag{17}$$

where all of these signals are $m \times 1$ vectors. These signals can be expressed as lag polynomials times the white noise process of fundamental exogenous shocks, $\varepsilon_{i,t}$, which has dimensionality $m_\varepsilon \geq m$ with positive definite variance matrix Σ_ε:

$$S_{Z,i,t} = S_Z(L)\varepsilon_{i,t} \qquad S_{X,i,t} = S_X(L)\varepsilon_{i,t} \tag{18}$$

The square-summable polynomial $S_Z(L)$ is a primitive of the model. But the polynomial $S_X(L)$ depends on equilibrium behavior and aggregation. Define the sum of the two polynomials as

$$Z_{i,t} = S(L)\varepsilon_{i,t} \equiv S_Z(L)\varepsilon_{i,t} + S_X(L)\varepsilon_{i,t} \tag{19}$$

The signal $Z_{i,t}$ is equivalent to two polynomials: $S(L)\varepsilon_{i,t}$ is a lag polynomial of fundamental shocks, while $Z(L)w_{i,t}$ is a lag polynomial of white noise innovations.

Aggregate variables affect the endogenous signal. Aggregate shocks must be defined before characterizing aggregate allocations. The shock $\varepsilon_{i,t}$ contains both aggregate and idiosyncratic dimensions. Suppose that there is a unit measure λ of agents i in the set $\mathcal{I}$. Assume that the idiosyncratic dimensions are mean zero in the population. The aggregate shock ε_t is defined

$$\varepsilon_t \equiv \int_{\mathcal{I}} \varepsilon_{i,t} d\lambda(i)$$

Then the average signal $Z_t \equiv \int_{\mathcal{I}} Z_{i,t} d\lambda(i)$ satisfies

$$Z_t = \int_{\mathcal{I}} S(L)\varepsilon_{i,t} d\lambda(i) = S(L)\varepsilon_t$$

[5] $(-B_{X1}^{-1}B_{X0})^{-1}$ is equivalent to Λ in Nimark (2017) Section 6.

because $Z(L)\varepsilon_{i,t}$ is linear in the sequence of shocks. Similarly, the aggregate endogenous vector $X_t \equiv \int_{\mathcal{I}} X_{i,t} d\lambda(i)$ satisfies

$$X_t = \int_{\mathcal{I}} X(L) w_{i,t} d\lambda(i) = \int_{\mathcal{I}} X(L) Z(L)^{-1} S(L) \varepsilon_{i,t} d\lambda(i) = X(L) Z(L)^{-1} S(L) \varepsilon_t \tag{20}$$

Finally, let the matrix I_ε denote the diagonal matrix with ones in dimensions corresponding to aggregate shocks and zeros elsewhere, so that

$$\varepsilon_t = I_\varepsilon \varepsilon_{i,t} \quad \forall i \in \mathcal{I} \tag{21}$$

The square-summable polynomial $P(L)$ determines how aggregate variables affect the endogenous signal. For example, it may include conditions relating aggregate allocations to idiosyncratic prices observed by the decision makers, or may include aggregate resource constraints or adding up constraints. $P(L)$ is a primitive of the model, and satisfies

$$S_{X,i,t} = [P(L) X_t]_+ \tag{22}$$

The right handside includes no idiosyncratic terms, so $S_{X,i,t}$ is the same for all agents; it is determined by macroeconomic aggregates X_t. The annihilation operator is included so that $P(L)$ might have terms associated with negative powers of L. This allows $P(L)$ to include aggregate expectational equations if needed.

The lag polynomial $S_X(L)$ is determined by combining equations (20), (21), and (22):

$$S_X(L)\varepsilon_{i,t} = [P(L) X(L) Z(L)^{-1} S(L) I_\varepsilon \varepsilon_{i,t}]_+ \tag{23}$$

The innovation polynomial $Z(L)$ and the signal polynomial $S(L)$ both produce the same series of $Z_{i,t}$, so they must have the same autocovariance function. Let the $m \times m$ matrix Γ_j denote the jth autocovariance of the signal $Z_{i,t}$. The fundamental shock $\varepsilon_{i,t}$ is a white noise process with variance Σ, so Γ_j satisfies

$$\Gamma_j = \sum_{k=0}^{\infty} S_k \Sigma S'_{k+j} \tag{24}$$

The innovation polynomial $Z(L)$ is the Wold decomposition of the signal polynomial $S(L)$, so its inverse $Z(L)^{-1}$ solves the Yule-Walker Equations:

$$\begin{pmatrix} \Gamma_0 & \Gamma_1 & \Gamma_2 & \dots \\ \Gamma_1 & \Gamma_0 & \Gamma_1 & \dots \\ \Gamma_2 & \Gamma_1 & \Gamma_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} -(Z^{-1})_1 \\ -(Z^{-1})_2 \\ -(Z^{-1})_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \vdots \end{pmatrix} \tag{25}$$

where the polynomial $Z(L)$ is normalized so that $Z_0 = I$.

3.1 Solution Algorithm

This section describes the algorithm to solve the endogenous information model.

Some notation is defined before outlining the algorithm. An arbitrary $n \times m$ square summable lag polynomial $Y(L) = \sum_{j=-\infty}^{\infty} Y_j L^j$ is a linear operator on a bi-infinite sequences of shocks.

Denote this operator as Y, which maps $\ell^2(\mathbb{R}^m)$ to $\ell^2(\mathbb{R}^n)$. The operator Y can be expressed as a bi-infinite dimensional block Toeplitz matrix $T(Y)$:

$$T(Y) \equiv \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \dots & Y_0 & Y_{-1} & Y_{-2} & Y_{-3} & \dots \\ \dots & Y_1 & Y_0 & Y_{-1} & Y_{-2} & \dots \\ \dots & Y_2 & Y_1 & Y_0 & Y_{-1} & \dots \\ \dots & Y_3 & Y_2 & Y_1 & Y_0 & \dots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

When $Y(L)$ is causal, so that it has $Y_j = 0$ for all $j < 0$, $T(Y)$ is lower block triangular. When $Y(L)$ is a constant matrix so that $Y_j = 0$ for all $j \neq 0$, then $T(Y)$ is a block diagonal matrix with Y_0 along the main block diagonal.

Algorithm 2 (Signal Operator Iteration) *Conjecture a square summable lag polynomial $S_0(L)$. Then proceed with iteration $n = 0$ by:*

1. *Find the autocovariance function $\Gamma_n(L)$ implied by $S_n(L)$ using equation* (24).
2. *Use $\Gamma_n(L)$ to solve the Yule-Walker equations* (25) *for $Z_n(L)$.*
3. *Given the signal process $Z_n(L)$, generate the polynomial $\tilde{Z}_n(L)$ by equation* (8).
4. *Calculate the policy function $X_n(L)$ from $\tilde{Z}_n(L)$ by Theorem 1.*
5. *Calculate the next signal polynomial $S_{n+1}(L)$ by combining equations* (19) *and* (23)*:*

$$S_{n+1}(L) = S_Z(L) + [P(L)X_n(L)Z_n^{-1}(L)S_n(L)I_\varepsilon]_+ \tag{26}$$

6. *If $||S_{n+1} - S_n||_{op}$ is sufficiently close to zero, conclude that $S(L) = S_{n+1}(L)$. Otherwise return to Step 1 with guess S_{n+1}.*

Theorem 8 states conditions on the model such that Signal Operator Iteration is a contraction mapping, and has a unique fixed point. Before stating the theorem, some further notation must be defined. For a given Wold decomposition $Z_n(L)$, let $C_{w,n}$ be a Cholesky decomposition of the corresponding innovation variance matrix $\Sigma_{w,n}$. Let C_ε denote the unique Cholesky decomposition of the fundamental shock variance Σ_ε, which is positive definite by assumption.

Lemma 3 *If $Z_n(L)$ is the Wold Representation of $S_n(L)$, then*

1. $||T(Z_n)T(C_{w,n})||_2 = ||T(S_n)T(C_\varepsilon)||_2$
2. $||T(C_{w,n}^{-1})T(Z_n^{-1})T(S_n)T(C_\varepsilon)||_2 = 1$

Proof. The jth autocovariance of $S_n(L)\varepsilon_{i,t}$ is given by

$$\Gamma_{n,j} = E[S_n(L)\varepsilon_{i,t}(S_n(L)\varepsilon_{i,t-j})']$$

The shocks $\varepsilon_{i,t}$ as i.i.d. across periods, so the autocovariance becomes

$$\Gamma_{n,j} = \sum_{k=0}^{\infty} S_{n,k}\Sigma_\varepsilon S'_{n,k+j}$$

By this equation, the block Toeplitz matrix of autocovariances $T(\Gamma_n)$ satisfies

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \dots & \Gamma_{n,0} & \Gamma_{n,1} & \Gamma_{n,2} & \dots \\ \dots & \Gamma_{n,1} & \Gamma_{n,0} & \Gamma_{n,1} & \dots \\ \dots & \Gamma_{n,2} & \Gamma_{n,1} & \Gamma_{n,0} & \dots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \dots & S_{n,0}C_\varepsilon & 0 & 0 & \dots \\ \dots & S_{n,1}C_\varepsilon & S_{n,0}C_\varepsilon & 0 & \dots \\ \dots & S_{n,2}C_\varepsilon & S_{n,1}C_\varepsilon & S_{n,0}C_\varepsilon & \dots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\times \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \dots & (S_{n,0}C_\varepsilon)' & (S_{n,1}C_\varepsilon)' & (S_{n,2}C_\varepsilon)' & \dots \\ \dots & 0 & (S_{n,0}C_\varepsilon)' & (S_{n,1}C_\varepsilon)' & \dots \\ \dots & 0 & 0 & (S_{n,0}C_\varepsilon)' & \dots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which can be written as

$$T(\Gamma_n) = T(S_nC_\varepsilon)T(S_nC_\varepsilon)'$$

so the largest eigenvalue of $T(\Gamma_n)$ is the largest singular value of $T(S_nC_\varepsilon)$.

By construction, $Z_n(L)w_{i,t}$ has the same autocovariance as $S_n(L)\varepsilon_{i,t}$, so $T(Z_nC_{w,n})$ satisfies

$$T(\Gamma_n) = T(Z_nC_{w,n})T(Z_nC_{w,n})'$$

Therefore, $T(Z_nC_{w,n}) = T(Z_n)T(C_{w,n})$ has the same largest singular value as $T(S_nC_\varepsilon) = T(S_n)T(C_\varepsilon)$. The largest singular value of a matrix is its Euclidean norm, proving result *1*.

The fundamental shock $\varepsilon_{i,t}$ and the innovation process $w_{n,i,t}$ implied by the Wold decomposition Z_n produce the same signal process, so they are related by

$$Z_n(L)w_{n,i,t} = S_n(L)\varepsilon_{i,t}$$

The Wold decomposition is invertible, so left multiplying by $C_{w,n}^{-1}Z_n(L)^{-1}$ yields

$$C_{w,n}^{-1}w_{n,i,t} = C_{w,n}^{-1}Z_n(L)^{-1}S_n(L)\varepsilon_{i,t}$$

This process is white noise with unit variance:

$$E[C_{w,n}^{-1}w_{n,i,t}(C_{w,n}^{-1}w_{n,i,t})'] = C_{w,n}^{-1}\Sigma_{w,n}C_{w,n}^{-1\prime} = I$$

Therefore the autocovariance Toeplitz matrix of $C_{w,n}^{-1}Z_n(L)^{-1}S_n(L)\varepsilon_{i,t}$ is the identity matrix, satisfying

$$I = T(C_{w,n}^{-1}Z_n(L)^{-1}S_n(L)C_\varepsilon)T(C_{w,n}^{-1}Z_n(L)^{-1}S_n(L)C_\varepsilon)'$$

The identity matrix has all unit eigenvalues, so the largest singular value of $T(C_{w,n}^{-1}Z_n(L)^{-1}S_n(L)C_\varepsilon)$ is 1. This is also its Euclidean norm, proving result *2*. ■

Some further notation is defined before the next lemma. Let $\mathcal{Q}_0$ denote the bi-infinite Toeplitz matrix with block row $\begin{pmatrix} Q_C\Lambda_C^{-k-1} & 0_{n_C\times n_S} \\ 0_{n_S\times n_C} & 0_{n_S\times n_S} \end{pmatrix} Q^{-1}(-B_{X1}^{-1}B_{Z0})$ for $k = 1, 2, ...$ starting at $k = 1$ on the main block diagonal. And let $\mathcal{Q}_1$ denote the bi-infinite Toeplitz matrix with block row $\begin{pmatrix} Q_C\Lambda_C^{-k-1} & 0_{n_C\times n_S} \\ 0_{n_S\times n_C} & 0_{n_S\times n_S} \end{pmatrix} Q^{-1}(-B_{X1}^{-1}B_{Z1})$ for $k = 1, 2, ...$ starting at $k = 1$ on the first upper block diagonal. Finally, define the lag polynomial $A_1(L) = -B_{X1}^{-1}(B_{Z0} + B_{Z1}L)$.

The lemma also uses the following properties of operators on Hilbert spaces:

Property 4 *The norm of the annihilator operator* $[\cdot]_+$*, which annihilates negative powers of* L*, is* $||[\cdot]_+||_{op} = 1$.

Property 4 holds because the annihilator operator sets some values (the non-causal terms) of a bi-infinite sequence of vectors to zero, but leaves the rest unchanged.

Property 5 *For any operator* Y *mapping* $\ell^2(\mathbb{R}^n)$ *to* $\ell^2(\mathbb{R}^m)$*,* $||Y||_{op} = ||T(Y)||_2$.

Property 5 holds because the Euclidean norm $||T(Y)||_2$ is equal to the largest singular value of $T(Y)$. Y is an operator on a Hilbert space, so the operator norm $||Y||_{op}$ is equal to the largest singular value of Y, and the largest singular values of $T(Y)$ and Y are equivalent.

Lemma 6 *Given Wold representation* $Z_n(L)$ *and Cholesky decomposition* $C_{w,n}$*,*

$$||X_n C_{w,n}||_{op} \leq ||\tilde{B}||_{op}(||A_1||_{op} + ||\mathcal{Q}_0 + \mathcal{Q}_1||_2 + ||B_{X1}^{-1}B_{Z1}||_2)||Z_n C_{w,n}||_{op} \tag{27}$$

Proof. Theorem 1 implies $X_n(L)C_{w,n} = [\tilde{B}(L)\tilde{Z}_n(L)C_{w,n}]_+$. Taking operator norms,

$$||X_n C_{w,n}||_{op} = ||[\tilde{B}\tilde{Z}_n C_{w,n}]_+||_{op}$$

and applying Property 4 yields

$$||X_n C_{w,n}||_{op} = ||\tilde{B}\tilde{Z}_n C_{w,n}||_{op} \tag{28}$$

which implies by the triangle inequality

$$||X_n C_{w,n}||_{op} \leq ||\tilde{B}||_{op}||\tilde{Z}_n C_{w,n}||_{op} \tag{29}$$

Per equation (8), the matrix representation of $\tilde{Z}(L)$ satisfies

$$T(\tilde{Z}) = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ ... & X_0 & 0 & 0 & ... \\ ... & -B_{X1}^{-1}(B_{Z0}Z_0 + B_{Z1}Z_1) & X_0 & 0 & ... \\ ... & -B_{X1}^{-1}(B_{Z0}Z_1 + B_{Z1}Z_2) & -B_{X1}^{-1}(B_{Z0}Z_0 + B_{Z1}Z_1) & X_0 & ... \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= - \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ ... & 0 & 0 & 0 & ... \\ ... & B_{X1}^{-1}(B_{Z0}Z_0 + B_{Z1}Z_1) & 0 & 0 & ... \\ ... & B_{X1}^{-1}(B_{Z0}Z_1 + B_{Z1}Z_2) & B_{X1}^{-1}(B_{Z0}Z_0 + B_{Z1}Z_1) & 0 & ... \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ ... & X_0 & 0 & 0 & ... \\ ... & 0 & X_0 & 0 & ... \\ ... & 0 & 0 & X_0 & ... \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which can be written for iteration n as

$$T(\tilde{Z}_n) = T(A_1)T(Z_n) + I_{UD}T(B_{X1}^{-1}B_{Z1}Z_{0,n}) + T(X_{0,n})$$

where I_{UD} is an infinite Toeplitz matrix with $n \times n$ identity matrices along the upper block diagonal, and $T(B_{X1}^{-1}B_{Z1}Z_{0,n})$ is a block diagonal matrix with $B_{X1}^{-1}B_{Z1}Z_{0,n}$ on the main block diagonal.

Right multiply by $T(C_{w,n})$ to get

$$T(\tilde{Z}_n)T(C_{w,n}) = T(A_1)T(Z_n)T(C_{w,n}) + I_{UD}T(B_{X1}^{-1}B_{Z1}Z_{0,n}C_{w,n}) + T(X_{0,n}C_{w,n})$$

which implies the inequality

$$||T(\tilde{Z}_n)T(C_{w,n})||_2 \leq ||T(A_1)T(Z_n)T(C_{w,n})||_2 + ||I_{UD}T(B_{X1}^{-1}B_{Z1}Z_{0,n}C_{w,n})||_2 + ||T(X_{0,n}C_{w,n})||_2 \tag{30}$$

Let I_1 denote the bi-infinite matrix with the $n \times n$ identity matrix in a single block on the main-diagonal, and zeros elsewhere. The matrix $(I_1T(B_{X1}^{-1}B_{Z1})T(Z_n)T(C_{w,n}))'I_1T(B_{X1}^{-1}B_{Z1})T(Z_n)T(C_{w,n})$ has $(B_{X1}^{-1}B_{Z1}Z_{0,n}C_{w,n})'B_{X1}^{-1}B_{Z1}Z_{0,n}C_{w,n}$ on a single main diagonal block and zeros elsewhere. $Z_{0,n} = I$ by construction so the largest singular value of $I_1T(B_{X1}^{-1}B_{Z1})T(Z_n)T(C_{w,n})$ is the largest singular value of $B_{X1}^{-1}B_{Z1}C_{w,n}$, implying

$$||I_{UD}T(B_{X1}^{-1}B_{Z1}Z_0C_{w,n})||_2 = ||I_1T(B_{X1}^{-1}B_{Z1})T(Z_n)T(C_{w,n})||_2 \tag{31}$$

Again let I_1 denote the bi-infinite matrix with the $n \times n$ identity matrix in a single block on the main-diagonal, and zeros elsewhere. Then the matrix $I_1(\mathcal{Q}_0 + \mathcal{Q}_1)T(Z_n)T(C_{w,n})$ has $X_{0,n}C_{w,n}$ on a single main diagonal block and zeros elsewhere. By the same logic as equation (31), the norm of the Toeplitz matrix $T(X_0C_{w,n})$ is given by

$$||T(X_{0,n}C_{w,n})||_2 = ||I_1(\mathcal{Q}_0 + \mathcal{Q}_1)T(Z_n)T(C_{w,n})||_2 \tag{32}$$

Substituting equations (31) and (32) into (30) yields

$$||T(\tilde{Z}_n)T(C_{w,n})||_2 \leq ||T(A_1)T(Z_n)T(C_{w,n})||_2...$$
$$+ ||I_1(\mathcal{Q}_0 + \mathcal{Q}_1)T(Z_n)T(C_{w,n})||_2 + ||I_1T(B_{X1}^{-1}B_{Z1})T(Z_n)T(C_{w,n})||_2$$

which implies

$$||T(\tilde{Z}_n)T(C_{w,n})||_2 \leq ||T(A_1)||_2||T(Z_n)T(C_{w,n})||_2...$$
$$+ ||I_1||_2||(\mathcal{Q}_0 + \mathcal{Q}_1)||_2||T(Z_n)T(C_{w,n})||_2 + ||I_1||_2||T(B_{X1}^{-1}B_{Z1})||_2||T(Z_n)T(C_{w,n})||_2$$

Substitute with the norms $||I_1||_2 = 1$ and $||T(B_{X1}^{-1}B_{Z1})||_2 = ||B_{X1}^{-1}B_{Z1}||_2$ and rearrange to get

$$||T(\tilde{Z}_n)T(C_{w,n})||_2 \leq (||T(A_1)||_2 + ||(\mathcal{Q}_0 + \mathcal{Q}_1)||_2 + ||T(B_{X1}^{-1}B_{Z1})||_2)||T(Z_n)T(C_{w,n})||_2$$

Finally, equation (29) and Property 5 imply

$$||X_nC_{w,n}||_{op} \leq ||\tilde{B}||_{op}||\tilde{Z}_nC_{w,n}||_{op} \leq ||\tilde{B}||_{op}(||A_1||_{op} + ||\mathcal{Q}_0 + \mathcal{Q}_1||_2 + ||B_{X1}^{-1}B_{Z1}||_2)||Z_nC_{w,n}||_{op}$$

which proves the lemma. ■

Lemma 7 *If C_ε is the Cholesky decomposition of the shock variance Σ_ε, then*

$$||S_{n+1}||_{op} \leq ||S_z||_{op} + ||P||_{op}||X_nC_{w,n}||_{op}||||C_\varepsilon^{-1}||_2 \tag{33}$$

Proof. $S_{n+1}(L)$ is calculated from equation (26). The operator norm of both sides of this equation is

$$||S_{n+1}||_{op} = ||S_z + [PX_nZ_n^{-1}S_nI_\varepsilon]_+||_{op}$$

and Property 4 implies

$$||S_{n+1}||_{op} \leq ||S_z||_{op} + ||PX_nZ_n^{-1}S_nI_\varepsilon||_{op} \tag{34}$$

The matrix form of $P(L)X_n(L)Z_n^{-1}(L)S_n(L)I_\varepsilon$ can be written as

$$T(P)T(X_n)T(Z_n^{-1})T(S_n)T(I_\varepsilon) = T(P)T(X_n)T(C_{w,n})T(C_{w,n}^{-1})T(Z_n^{-1})T(S_n)T(C_\varepsilon)T(C_\varepsilon^{-1})T(I_\varepsilon)$$

because the $T(C_{w,n}^{-1})$ is the inverse of $T(C_{w,n})$, and$T(C_\varepsilon^{-1})$ is the inverse of $T(C_\varepsilon)$. Taking norms yields the inequality

$$\begin{aligned}&||T(P)T(X_n)T(Z_n^{-1})T(S_n)T(I_\varepsilon)||_2 ...\\ &\qquad \leq ||T(P)||_2||T(X_n)T(C_{w,n})||_2||T(C_{w,n}^{-1})T(Z_n^{-1})T(S_n)T(C_\varepsilon)||_2||T(C_\varepsilon^{-1})||_2||T(I_\varepsilon)||_2\end{aligned}$$

$T(I_\varepsilon)$ has ones in some diagonal entries and zeros elsewhere, so $||T(I_\varepsilon)||_2 = 1$. And by Lemma 3, $||T(C_{w,n}^{-1})T(Z_n^{-1})T(S_n)T(C_\varepsilon)||_2 = 1$, so the inequality becomes

$$||T(P)T(X_n)T(Z_n^{-1})T(S_n)T(I_\varepsilon)||_2 \leq ||T(P)||_2||T(X_n)T(C_{w,n})||_2||T(C_\varepsilon^{-1})||_2$$

Applying property 5 and substituting into (34) implies

$$||S_{n+1}||_{op} \leq ||S_z||_{op} + ||P||_{op}||X_nC_{w,n}||_{op}||||C_\varepsilon^{-1}||_2$$

which completes the proof.

■

Here some final notation is defined before presenting the Signal Operator Contraction theorem. Define the operator $\mathcal{B}$ as applying steps *1-5* of Algorithm 2, so that guesses of the signal operator S_n and S_{n+1} are related by

$$S_{n+1} = \mathcal{B}S_n$$

Let $\mathcal{S}$ denote the set of operators that map $\ell^2(\mathbb{R}^m)$ to $\ell^2(\mathbb{R}^n)$. Define the distance on this set $d(S_a, S_b)$ as

$$d(S_a, S_b) \equiv |(||S_a||_{op} - ||S_b||_{op})|$$

where S_a and S_b are members of $\mathcal{S}$ and $||\cdot||_{op}$ is the relevant operator norm.

Theorem 8 (Signal Operator Contraction) *Let*

$$\beta \equiv ||P||_{op}||\tilde{B}||_{op}(||A_1||_{op} + ||(\mathcal{Q}_0 + \mathcal{Q}_1)||_2 + ||B_{X1}^{-1}B_{Z1}||_2)||C_\varepsilon||_2||C_\varepsilon^{-1}||_2$$

If $\beta < 1$ then $\mathcal{B}$ is a contraction on $\mathcal{S}$ with modulus β, and Signal Operator Iteration has a unique fixed point $S = \mathcal{B}(S)$.

Proof. $\mathcal{B}$ is an operator mapping $\mathcal{S}$ to $\mathcal{S}$. Consider $S_n \in \mathcal{S}$ and $S_{n+1} = \mathcal{B}(S_n)$. By Lemma 3 and Property 5,

$$||Z_nC_{w,n}||_{op} = ||S_nC_\varepsilon||_{op}$$

Which implies the inequality $||Z_nC_{w,n}||_{op} \leq ||S_n||_{op}||C_\varepsilon||_{op}$. Combining this with Lemmas 6 and 7 and Property 5 relates the operator norms by

$$||S_{n+1}||_{op} \leq ||S_z||_{op} + ||P||_{op}||\tilde{B}||_{op}(||A_1||_{op} + ||\mathcal{Q}_0 + \mathcal{Q}_1||_2 + ||B_{X1}^{-1}B_{Z1}||_2)||S_n||_{op}||C_\varepsilon||_2||C_\varepsilon^{-1}||_2 \tag{35}$$

By assumption, S_z is an operator mapping $\ell^2(\mathbb{R}^m)$ to $\ell^2(\mathbb{R}^n)$, while P, $\tilde{B}$, and A_1 are operators mapping $\ell^2(\mathbb{R}^n)$ to $\ell^2(\mathbb{R}^n)$.

Now consider any two operators S_a and S_b in $\mathcal{S}$. Inequality (35) implies that $\mathcal{B}(S_a)$ and $\mathcal{B}(S_b)$ satisfy

$$||\mathcal{B}(S_a)||_{op} - ||\mathcal{B}(S_b)||_{op} \leq ... \\ ||P||_{op}||\tilde{B}||_{op}(||A_1||_{op} + ||(\mathcal{Q}_0 + \mathcal{Q}_1)||_2 + ||B_{X1}^{-1}B_{Z1}||_2)(||S_a||_{op} - ||S_b||_{op})||C_\varepsilon||_2||C_\varepsilon^{-1}||_2 \quad (36)$$

Substituting for β and taking the absolute value of both sides of (36) gives

$$|(||\mathcal{B}(S_a)||_{op} - ||\mathcal{B}(S_b)||_{op})| \leq \beta|(||S_a||_{op} - ||S_b||_{op})|$$

so by the contraction mapping theorem, $\mathcal{B}$ is a contraction on $\mathcal{S}$ with modulus β, and has a unique fixed point in $\mathcal{S}$. ∎

4 A Local Information Model

The solution method is applied to a model of local information. The economy is made up of "islands" in the style of Lucas and Prescott (1974). Firms and households observe prices and quantities on each island, but not the aggregate state of the economy. They get some information by observing local market conditions, but there are more shocks than informative signals, so the aggregate state is not revealed. The shocks that confound information map directly into observable quantities in the data: idiosyncratic productivity shocks, sectoral demand shocks, and monetary shocks. In equilibrium, money is non-neutral because firms cannot perfectly distinguish between monetary shocks and real aggregate shocks.

4.1 Households

There is a set of islands $\mathcal{I}$ indexed by i. On each island, there is a unit measure $\lambda(i) = 1$ identical and infinitely lived households.

The island i representative household's preferences over current and future consumption are represented by the utility function

$$E_{i,t}\left[\sum_{s=0}^{\infty}\beta^s\frac{C_{i,t+s}^{1-\gamma}-1}{1-\gamma}\right] \quad (37)$$

where $C_{i,t}$ is the household's consumption in period t, β is their discount factor, and γ is their coefficient of relative risk aversion. The expectation operator $E_{i,t}$ is conditional on the representative household i's information set $\Omega_{i,t}$.

Households earn two types of income. They inelastically supply one unit of labor on their island, for which they are paid nominal wage $W_{i,t}$. The also own the capital on their island, $K_{i,t}$, which they rent to firms at nominal rental rate $R_{K,i,t}$.

Households spend their income on two types of goods. They purchase generic output goods from an economy-wide market at aggregate price level P_t. This generic good can be used for either consumption $C_{i,t}$ or investment $I_{i,t}$. Therefore their budget constraint is

$$W_{i,t} + R_{K,i,t}K_{i,t} = P_tC_{i,t} + P_tI_{i,t} \quad (38)$$

Investment faces an adjustment cost $\varphi(\frac{I_{i,t}}{K_{i,t}})$ which affects the productivity of investment goods at producing new capital. A household owning $K_{i,t}$ capital and investing $I_{i,t}$ faces the law of motion:

$$K_{i,t+1} = I_{i,t}(1 - \varphi(\frac{I_{i,t}}{K_{i,t}})\frac{K_{i,t}}{I_{i,t}}) + (1-\delta)K_{i,t} \tag{39}$$

where the adjustment cost function satisfies $\varphi(\delta) = 0$, $\varphi'(\delta) = 0$ and $\varphi''(\delta) > 0$.

The household's problem is to choose sequences of $C_{i,t}$, $I_{i,t}$ and $K_{i,t+1}$ to maximize (37) subject to the budget constraint (38) and law of motion (39). The solution to this problem is characterized by an Euler equation (40):

$$\frac{1}{1-\varphi'(\frac{I_{i,t}}{K_{i,t}})} = E_{i,t}\left[\beta\left(\frac{C_{i,t}}{C_{i,t+1}}\right)^{\gamma}\left(\frac{R_{K,i,t+1}}{P_{t+1}} + \frac{1}{1-\varphi'(\frac{I_{i,t+1}}{K_{i,t+1}})}\left(\varphi'(\frac{I_{i,t+1}}{K_{i,t+1}})\frac{I_{i,t+1}}{K_{i,t+1}} - \varphi(\frac{I_{i,t+1}}{K_{i,t+1}}) + 1 - \delta\right)\right)\right] \tag{40}$$

where expectations $E_{i,t}$ are conditional on representative household i's information set $\Omega_{i,t}$. On the left-hand side of the Euler equation, $\frac{1}{1-\varphi'(\frac{N_{i,t}}{K_{i,t}})}$ is Tobin's Q, the marginal cost of an additional unit of capital for firms in market i at time t. On the right-hand side, households discount the real return on their capital, plus the marginal units of capital they carry over.

4.2 Firms

There are two types of firms in the economy. There are intermediate goods firms that each operate on an island indexed by i, and there are final goods firms that aggregate the intermediate goods into a final output good in an economy-wide market.

Final goods firms aggregate specialized goods Y_i of type $i \in \mathcal{I}$ with a CES production function,

$$Y_t = \left(\int_{i\in\mathcal{I}} G_{i,t}^{\frac{1}{\eta}} Y_{i,t}^{\frac{\eta-1}{\eta}} d\lambda(i)\right)^{\frac{\eta}{\eta-1}} \tag{41}$$

with $\eta \neq 1$. $G_{i,t}$ is a good-specific stochastic shock and is i.i.d across types.

The final goods sector is perfectly competitive, so the price of output P_t is given by the CES price aggregator,

$$P_t = \left(\int_{i\in\mathcal{I}} G_{i,t} P_{i,t}^{1-\eta} d\lambda(i)\right)^{\frac{1}{1-\eta}} \tag{42}$$

and final goods firms' demand for intermediates is given by the CES demand function

$$P_{i,t} = P_t(\frac{G_{i,t}Y_t}{Y_{i,t}})^{\frac{1}{\eta}} \tag{43}$$

Intermediate goods firms are perfectly competitive and have constant returns. The representative firm on island i uses specialized capital $K_{i,t}$ and labor $L_{i,t}$ with stochastic productivity $A_{i,t}$ to produce output $Y_{i,t}$ by

$$Y_{i,t} = A_{i,t}K_{i,t}^{\alpha}L_{i,t}^{1-\alpha} \tag{44}$$

Firms rent capital at nominal rental rate $R_{K,i,t}$ and hire labor at nominal wage $W_{i,t}$ from the households on island i. They sell their output at price $P_{i,t}$. The representative firm chooses inputs to maximize their profits, which implies that labor and capital demands for island i are given by

$$P_{i,t}\alpha\frac{Y_{i,t}}{K_{i,t}} = R_{K,i,t} \qquad P_{i,t}(1-\alpha)\frac{Y_{i,t}}{L_{i,t}} = W_{i,t} \tag{45}$$

4.3 Money and Information

Money's only role is to determine the price level. Specifically, money M_t is a stochastic process for nominal aggregate output:

$$M_t = P_t Y_t \tag{46}$$

Households observe the aggregate price level because they buy goods for consumption and investment at price P_t, but they cannot observe M_t directly, so they cannot directly infer Y_t. This is the information friction that allows monetary shocks to affect the real economy.

The money supply is determined by a stochastic money supply rule, in which money growth depends on aggregate output and a stochastic term $\tilde{\mu}_t$.

$$\ln M_t = \phi_Y \ln Y_t + \tilde{\mu}_t \tag{47}$$

The rule might be set by a monetary authority that observes the aggregate state of the economy, and for which the parameter ϕ_Y determines the dependence of the money supply on real aggregate output. A value of $\phi_Y = 1$ implies that money supply will move one-for-one with aggregate output, as if the monetary authority is targeting a particular price level (or a particular inflation rate if $\tilde{\mu}_t$ has a trend component.) The stochastic shock $\tilde{\mu}_t$ might represent error with which the monetary authority measures Y_t, or it could represent other stochastic factors it considers when setting money supply.

The price level is one of three endogenously-noisy signals that inform islands about the state of the aggregate economy. The other signals are demand for specialized goods and island-specific productivity.

Firms can see demand for their goods, which is a noisy signal of the aggregate output level Y_t. They observe local prices and quantities, and the aggregate price level P_t, but cannot directly observe whether changes in real demand for their specialized goods $Y_{i,t}$ is driven by aggregate output Y_t or by the sector-specific shock $G_{i,t}$. Through the sectoral demand equation (43), they observe the value of the demand signal

$$H_{i,t} \equiv (G_{i,t} Y_t)^{\frac{1}{\eta}} \tag{48}$$

but not the individual components.

In logs, productivity $\ln A_{i,t}$ is the sum of an aggregate component $\ln A_t$ and a mean zero idiosyncratic component $\ln \hat{A}_{i,t}$ satisfying

$$\ln A_{i,t} = \ln A_t + \ln \hat{A}_{i,t} \tag{49}$$

Firms cannot observe aggregate productivity directly, but must make inference based on their idiosyncratic productivity.

The firm's information set $\Omega_{i,t}$ includes all of the local endogenous variables on island i, plus the aggregate price level P_t, and demand signal $H_{i,t}$, and the information set evolves by

$$\Omega_{i,t} = \{\Omega_{i,t-1}, P_{i,t}, Y_{i,t}, A_{i,t}, K_{i,t+1}, N_{i,t}, W_{i,t}, L_{i,t}, D_{i,t}, H_{i,t}, P_t\} \tag{50}$$

Many of these quantities are redundant in equilibrium. Together, firms observe three noisy signals of the aggregate state: productivity $A_{i,t}$, the demand signal $H_{i,t}$, and the price level P_t. Because there are four shocks, the aggregate state of the economy cannot be revealed if none of the signals are perfectly collinear.

The novel characteristic of the information structure is that signals are endogenous and their noise is measureable in the data. Households see prices and quantities, which inform their forecasts. This differs from papers such as Melosi (2016) and Woodford (2003), where agents observe exogenous noisy signals of aggregate shocks.

4.4 Equilibrium Definition

Given infinite sequences of exogenous variables $\{G_{i,t}, \ln \hat{A}_{i,t}, \ln A_t, M_t\}$ for all $i \in \mathcal{I}$, a competitive equilibrium in this economy consists of infinite sequences of prices, $\{P_{i,t}, P_t, W_{i,t}, R_{K,i,t}\}$ for all $i \in \mathcal{I}$; allocations $\{C_{i,t}, I_{i,t}, K_{i,t}, L_{i,t}, Y_{i,t}, Y_t\}$ for all $i \in \mathcal{I}$; and information sets $\Omega_{i,t}$ for all $i \in \mathcal{I}$ such that:

1. Households maximize utility (37), subject to the constraints (38) and (39)
2. Intermediate firms choose allocations to maximize profits, satisfying the production function (44) and factor demands (45).
3. Final goods firms choose allocations to maximize profits, satisfying the production function (41) and input demands (43).
4. Money determines the aggregate price level by (46)
5. Firm productivity is given by (49)
6. Information sets evolve by (50)
7. The labor market clears: $L_{i,t} = 1$ for all $i \in \mathcal{I}$

4.5 Linearization

The model must be put in a linear form that can be solved by Signal Operator Iteration.

4.5.1 Linear Equilibrium Conditions

First, the equilibrium conditions must be linearized. Let lower case variables denote log deviations from the deterministic steady state. By combining equations, household i's choice variables can be reduced to one control $i_{i,t}$ and one state $k_{i,t}$. These quantities are determined by two linear equilibrium conditions expressed in terms of the log-linearized observable signals: productivity $a_{i,t}$, demand $h_{i,t}$, and inflation π_t.

Log linearizing the household's Euler equation (40) and substituting in with the capital demand equation, sectoral demand equation, and budget constraint yields

$$\bar{\varphi}(i_{i,t} - k_{i,t}) = E_{i,t}\left[\gamma\left(\frac{\bar{Y}}{\bar{C}}(a_{i,t} + \alpha k_{i,t}) - \frac{\bar{I}}{\bar{C}}i_{i,t} - \frac{\bar{Y}}{\bar{C}}(a_{i,t+1} + \alpha k_{i,t+1}) + \frac{\bar{I}}{\bar{C}}i_{i,t+1}\right)\right] + ...$$
$$E_{i,t}\left[\beta\alpha\frac{\bar{Y}}{\bar{K}}(h_{i,t+1} + (1 - \frac{1}{\eta})a_{i,t+1} + (\alpha(1 - \frac{1}{\eta}) - 1)k_{i,t+1}) + \beta(1-\delta)(1+\delta)\bar{\varphi}(i_{i,t+1} - k_{i,t+1})\right] \quad (51)$$

where $\beta \equiv \frac{1}{1+\rho}$, $\bar{\varphi} \equiv \varphi''(\delta)\delta$, and variables with overlines, e.g. $\bar{Y}$, denote stead state levels. This linearized Euler equation is derived explicitly in Appendix A.

The second equilibrium condition for island i is the linearized law of motion for capital:

$$k_{i,t+1} \approx \frac{\bar{I}}{\bar{K}}i_{i,t} - (1-\delta)k_{i,t} \quad (52)$$

With these two linear equations, the model can be expressed in matrix form, corresponding to equation (1). The endogenous vector $X_{i,t}$ and the exogenous vector (from the perspective of island i) $Z_{i,t}$ are given by

$$X_{i,t} = \begin{pmatrix} i_{i,t} \\ k_{i,t} \end{pmatrix} \qquad Z_{i,t} = \begin{pmatrix} a_{i,t} \\ h_{i,t} \\ \pi_t \end{pmatrix} \tag{53}$$

while the coefficient matrices encoding equations (51) and (52) are given by

$$B_{X0} = \begin{pmatrix} -\bar{\varphi} - \gamma\frac{\bar{I}}{\bar{C}} & \bar{\varphi} + \gamma\alpha\frac{\bar{Y}}{\bar{C}} \\ \frac{\bar{I}}{\bar{K}} & -(1-\delta) \end{pmatrix} \tag{54}$$

$$B_{X1} = \begin{pmatrix} \beta(1-\delta)(1+\delta)\bar{\varphi} + \gamma\frac{\bar{I}}{\bar{C}} & -\beta(1-\delta)(1+\delta)\bar{\varphi} - \gamma\alpha\frac{\bar{Y}}{\bar{C}} + \beta\alpha\frac{\bar{Y}}{\bar{K}}(\alpha(1-\frac{1}{\eta})-1) \\ 0 & -1 \end{pmatrix} \tag{55}$$

$$B_{Z0} = \begin{pmatrix} \gamma\frac{\bar{Y}}{\bar{C}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B_{Z1} = \begin{pmatrix} -\gamma\frac{\bar{Y}}{\bar{C}} & \beta\alpha\frac{\bar{Y}}{\bar{K}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{56}$$

Inflation π_t does not show up in the linearized equilibrium conditions, which are expressed in real terms. Inflation serves only as a noisy signal of the aggregate state.

4.5.2 Linear Signal Formation

Next, the three observable signals of the aggregate economy ($a_{i,t}$, $h_{i,t}$, π_t) must be linearly expressed in terms of the exogenous variables and endogenous aggregates.

By assumption, log-linearized productivity is given by

$$a_{i,t} = \hat{a}_{i,t} + a_t \tag{57}$$

The linearized demand signal (48) is given by

$$h_{i,t} = \frac{1}{\eta}g_{i,t} + \frac{1}{\eta}(a_t + \alpha k_t) \tag{58}$$

where log-linearized aggregate output is replaced by $y_t = a_t + \alpha k_t$.

Inflation is used instead of the price level, so that the money supply can have a unit root while ensuring that the linearized system remains stationary. Accordingly, inflation is given by

$$\pi_t = (\phi_Y - 1)(a_t - a_{t-1} + \alpha(k_t - k_{t-1})) + \mu_t \tag{59}$$

where the exogenous term μ_t is given by $\mu_t \equiv \tilde{\mu}_t - \tilde{\mu}_{t-1}$, the first difference of the money supply shock in Equation (47).

These three linear equations determining the signals are encoded into the lag polynomials given by the following matrix equation:

$$Z_{i,t} = S_Z(L)\varepsilon_t + P(L)X_t \tag{60}$$

where ε_t is the 4×1 vector of fundamental innovations ot the exogenous variables and $S_Z(L)$ is a 3×4 lag polynomial encoding the time series properties of the exogenous shocks. To produce $Z_{i,t} = \begin{pmatrix} a_{i,t} & h_{i,t} & \pi_t \end{pmatrix}'$, these polynomials must satisfy Equations (57), (58), and (59):

$$S_Z(L)\varepsilon_t = \begin{pmatrix} \hat{a}_{i,t} + a_t \\ \frac{1}{\eta}a_t + \frac{1}{\eta}g_{i,t} \\ (\phi_Y - 1)(a_t - a_{t-1}) + \mu_t \end{pmatrix} \quad P(L)X_t = \begin{pmatrix} 0 \\ \frac{1}{\eta}\alpha k_t \\ (\phi_Y - 1)\alpha(k_t - k_{t-1}) \end{pmatrix} \tag{61}$$

This implies that the lag polynomial $P(L) = \sum_{j=0}^{\infty} P_j L^j$ is given by

$$P_0 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\eta}\alpha \\ 0 & (\phi_Y - 1)\alpha \end{pmatrix} \quad P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -(\phi_Y - 1)\alpha \end{pmatrix} \tag{62}$$

with $P_j = 0$ otherwise.

4.6 Calibration

The local information model is calibrated to resemble the US economy. Values for preference and production parameters are chosen, while the time series processes for fundamental shocks are estimated from national accounting and industry-level data.

Two of the chosen parameters are selected to guarantee uniqueness of the equilibrium: the elasticity of substitution η, and the output coefficient in money growth ϕ_Y. To understand how these parameters can guarantee uniqueness, consider the matrices in the lag polynomial $P(L)$, which controls how endogenous variables feed back into the signal process (Equation (62)). η can be set arbitrarily large, and ϕ_Y can be set arbitrarily close to one, to ensure that the terms in these matrices becomes arbitrarily close to zero. Accordingly, choosing parameter values of η and ϕ_Y can insure that the operator norm $||P||_{OP}$ is arbitrarily close to zero, and $||P||_{OP}$ is a coefficient in the definition of the *Signal Operator Iteration* modulus, so convergence can be guaranteed.

Intuitively, the parameters ϕ_Y and γ control the information content in agents' endogenous signals of the aggregate state. For example, ϕ_Y close to one makes the inflation rate an extremely noisy signal of aggregate output growth. The effect of η is less straightforward, because larger η simultaneously increases $||B||_{OP}$. I set $\phi_Y = 0.99$ and $\eta = 30$ in the baseline, but also consider alternative values to illustrate the impact of endogenous information content on the economy's dynamics. The other chosen parameters are set to standard values.[6] The implied modulus of the algorithm for the baseline calibration is 0.91, so convergence and uniqueness are guaranteed by Theorem 8.

The nominal shock process is estimated from the consumption price deflator in the US national accounts. The real shock processes are estimated from the TFP series in the US KLEMS industry-level data (Jorgenson, Ho, and Samuels, 2012). I interpret an island i to be a KLEMS industry top-level industry, of which there are 29. In the model, the idiosyncratic demand shock $G_{i,t}$ is identified by

$$G_{i,t} = \frac{Y_{i,t}}{Y_t}\left(\frac{P_{i,t}}{P_t}\right)^{\eta}$$

When mapping this object to the data, $\frac{Y_{i,t}}{Y_t}$ uses the quantity index of industry i relative to the total output quantity index, and the relative price level $\frac{P_{i,t}}{P_t}$ uses the price index of industry i relative to the the output price index.

In all cases, the series are logged and detrended using the Christiano-Fitzgerald filter (Christiano and Fitzgerald, 2003), removing only the highest and lowest frequencies: below 2 years and above 50 years. Then AR(1) processes are estimated for the detrended data. The estimated parameters are reported in table 1.

[6] The only unusual value among the remaining chosen parameters is risk aversion $\gamma = 0.5$, which is on the low end of calibrated values in macro models (as in Lucas (1990)). Lower risk aversions (i.e. higher intertemporal elasticities of substitution) reduce the modulus of the algorithm as well.

Parameter	Interpretation	Value	Justification
β	Discount factor	0.95	5% Annual real return
δ	Depreciation factor	0.1	Standard value
α	Capital share	0.33	Standard value
γ	Risk aversion	.5	Standard value
η	Elasticity of substitution	30	Guaranteed Uniqueness
ϕ_y	Output Coefficient in Money Rule	0.99	Guaranteed Uniqueness
$\rho_{\hat{a}}$	Persistence of idiosyncratic technology shock	0.80	US KLEMS
ρ_a	Persistence of aggregate technology shock	0.78	US KLEMS
ρ_g	Persistence of idiosyncratic demand shock	0.83	US KLEMS
ρ_m	Persistence of nominal shock	0.94	US National Accounts
$\sigma_{\hat{a}}$	Standard deviation of idiosyncratic technology shock	0.08	US KLEMS
σ_a	Standard deviation of aggregate technology shock	0.01	US KLEMS
σ_g	Standard deviation of idiosyncratic demand shock	0.20	US KLEMS
σ_m	Standard deviation of nominal shock	0.05	US National Accounts

Table 1: Illustrative calibration

These shock processes control how well firms can estimate the aggregate state from their observed signals. An advantage of the local information model is that the noise-inducing shock processes to be estimated directly from the data. In models where the noise is exogenous, it is not directly empirically observable, so the time series properties of the noise cannot be informed by micro data.

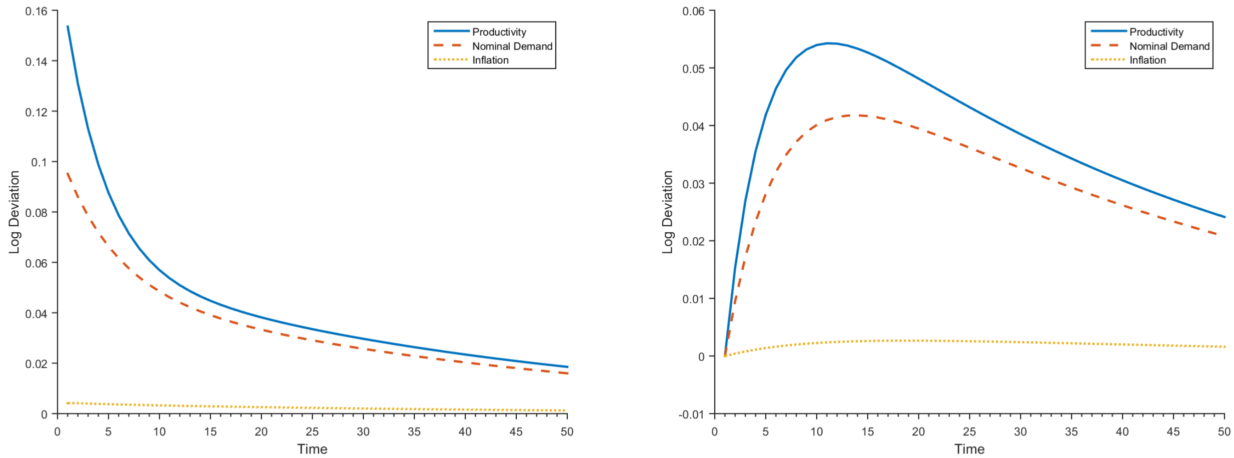
4.7 Equilibrium Dynamics

Agents observe 3 noisy signals: their productivity, real demand for their goods, and the inflation rate. But there are 4 fundamental shocks, so these signals cannot be perfectly revealing. Figure 1 plots islands' impulse responses to innovations in each signal.

First, a productivity innovation has classical effects; islands consume and invest more, increasing their capital and output in future periods. But they are uncertain about whether their productivity shock is aggregate or idiosyncratic.

Second, a positive demand innovation can be driven either by an idiosyncratic increase in demand for a firms' goods, or a positive aggregate productivity shock that raises aggregate output. These two possibilities lead to different investment behaviors. If agents knew they faced an idiosyncratic demand shock, they would sell more goods and invest less, because they expect relative prices for their own goods to decline next period, as the idiosyncratic shock decays over time. However if agents knew that the increase in demand was driven by an aggregate productivity shock, they would invest more expecting higher future demand - given that aggregate output has a hump-shaped response to aggregate productivity shocks - although effect this would partially be offset by the wealth effect to consumers.

Third, an inflation innovation has no direct real effects on an agent's equilibrium conditions; it only affects their expectations. Inflation is a noisy signal of aggregate output growth, so it changes agents' forecasts for future productivity and demand. Given the same level of idiosyncratic productivity and demand, an agent seeing unexpectedly high inflation will infer that their observed demand is likely to be driven by relatively higher productivity and relatively lower idiosyncratic demand, and so will increase investment. In the baseline calibration, ϕ_Y is close to one, so inflation is an very uninformative signal about the aggregate state, and the investment response to an inflation innovation is small.

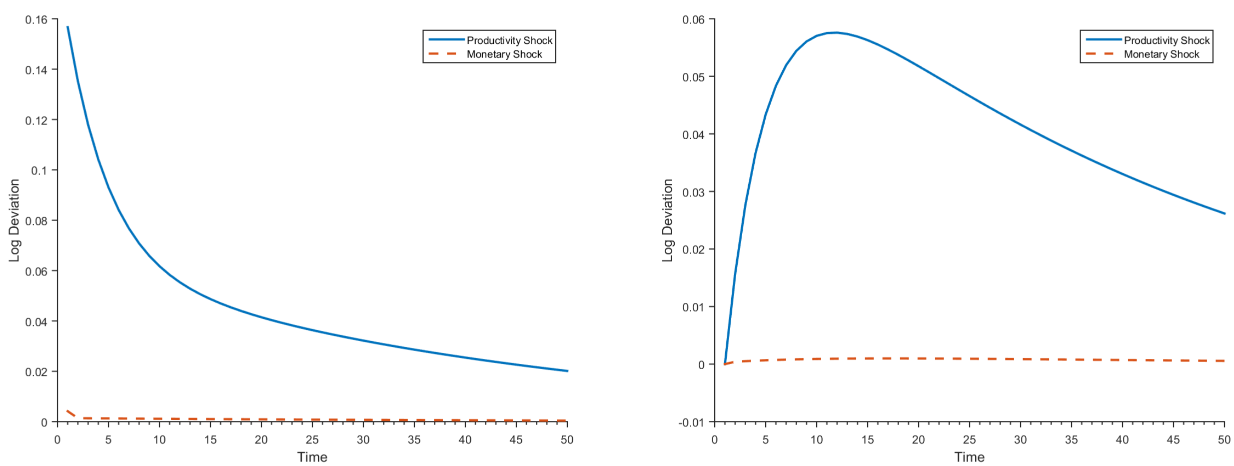


(a) Investment (b) Capital

Figure 1: Innovation Impulse Responses: Baseline

In the aggregate, idiosyncratic demand and productivity shocks sum to zero, so only aggregate productivity and monetary shocks have aggregate effects. Figure 2 plots the impulse responses of consumption and capital to aggregate productivity and monetary shocks. A productivity shock resembles the standard response in an RBC model, although agents' response is different than if they could observe the shock directly.

In contrast, the monetary shock is non-classical. A monetary shock has a real effect, because agents cannot perfectly distinguish its effect on inflation from a possible productivity shock. In response, aggregate investment increases, but only slightly, as the confounding inflationary signal does not contain much information. However, when the inflationary signal contains *more* information, the nominal shock is *more* distortionary. I demonstrate this effect in the next section.



(a) Investment (b) Capital

Figure 2: Aggregate Impulse Responses: Baseline

4.8 Effects of Increasing Information in the Endogenous Signal

In the baseline calibration, inflation is relatively uninformative about the aggregate state fo the economy. In this section, I increase the precision of this endogenous signal and show that it increases the real effect of nominal shocks.

I modify the baseline calibration by setting $\phi_Y = 0.5$. This increases the weight placed on aggregate output growth in the determination of the inflation rate (Equation (59)). [7] The impulse responses to innovations and aggregate shocks are plotted in Figure 3.

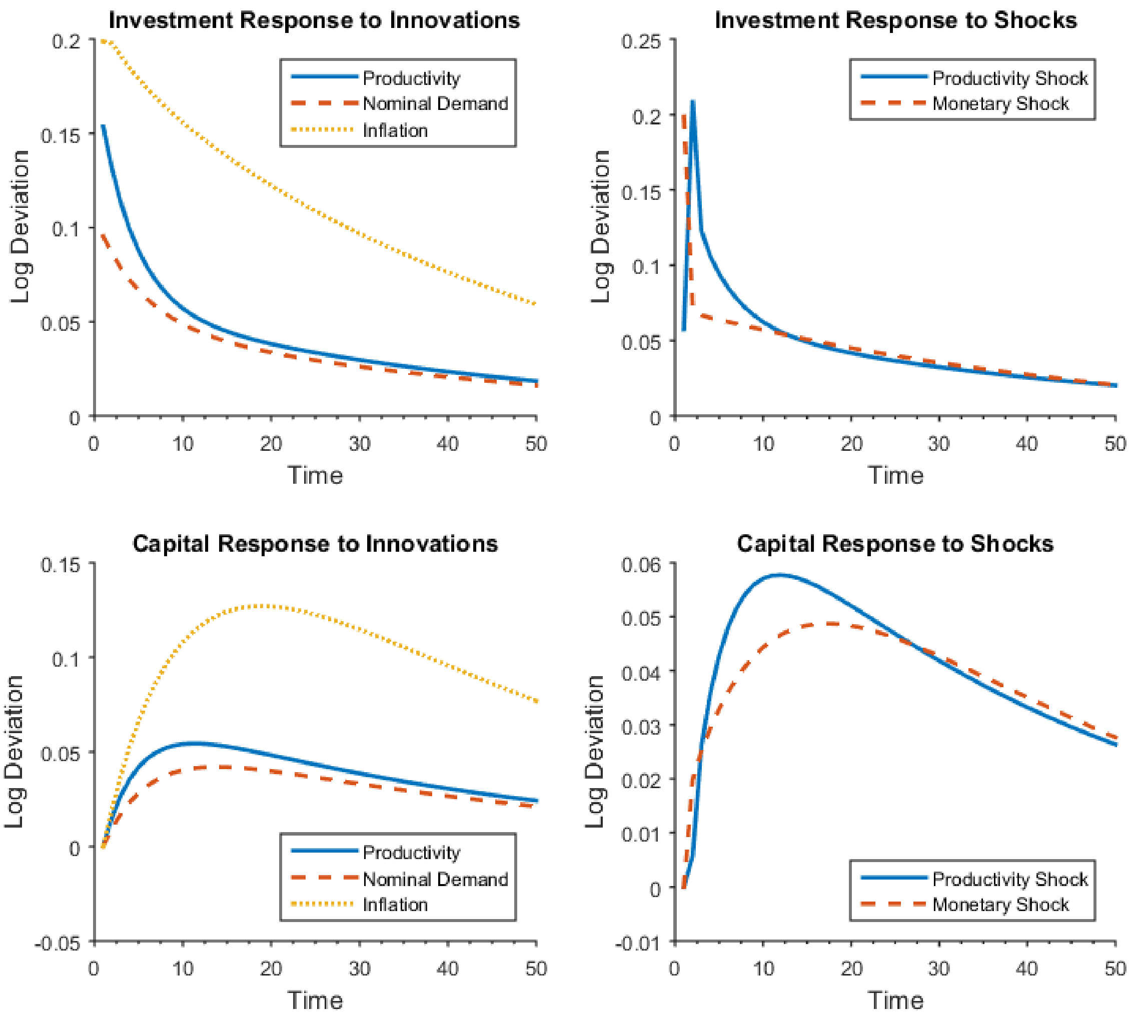


Figure 3: Impulse Responses: $\phi_Y = 0.5$

When agents see a positive inflation innovation with $\phi_Y = 0.5$, they can infer that a positive aggregate productivity shock was likely. They increase investment, because they expect demand for their island's output to rise as the economy booms in response to the aggregate productivity improvement. Households quickly increase their certainty about whether the inflation innovation

[7] The *Signal Operator Iteration* modulus also increases above one, so uniqueness and convergence are no longer guaranteed by Theorem 8, but in practice I have found no other equilibria, nor initial guesses of signal processes that did not converge.

was driven by real or nominal shocks. Changes in following periods to an island's productivity and demand inform the household about the aggregate state, although they never learn it with certainty. Accordingly, the aggregate effect of a monetary shock decays rapidly after the first period as agents learn, while the effect of a productivity shock increases after the first period when agents become more certain that observed inflation was driven by real forces.

When the inflation signal contains more information about the aggregate state of the economy, then nominal shocks have stronger real effects. Figure 4 plots the cumulative effect of an aggregate monetary shock on investment for different values of ϕ_Y. When $\phi_Y = 1$, inflation reveals no information about the level of real aggregate output, and nominal shocks have no real effects. When ϕ_Y deviates from 1 in either direction, the model exhibits monetary non-neutrality. As the inflationary signal becomes more informative about the aggregate state of the economy, nominal shocks have larger real effects in absolute value.

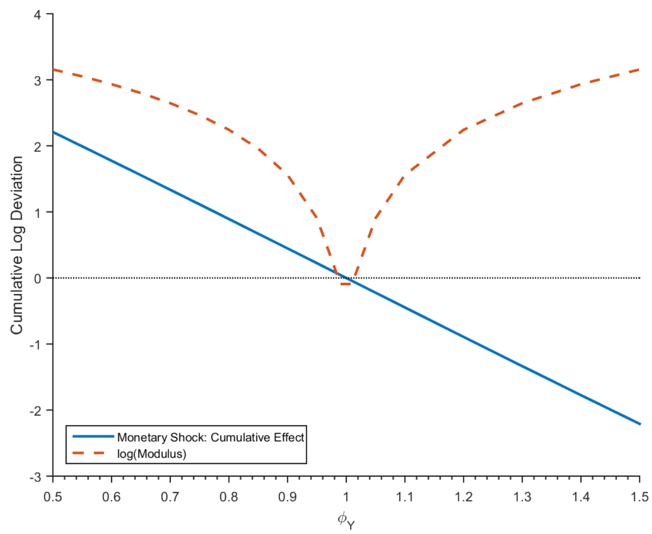


Figure 4: Effects of Noise in the Inflation Signal

The convergence properties of the algorithm are also affected by the information content of inflation. Figure 4 plots the log modulus of the algorithm, which is increasing in the distance between ϕ_Y and 1. As inflation becomes more informative, the feedback from the real economy to the endogenous signal increases. As a result, the speed of convergence of the Signal Operator Iteration slows. Convergence still occurs for all of these calibrations despite moduli above one, suggesting that there might be less restrictive sufficient criteria to insure uniqueness and convergence than those of Theorem 8.

5 Conclusion

This paper introduced a general method for solving macroeconomic models with endogenous information: *Signal Operator Iteration*. It demonstrated how to apply the algorithm and gave conditions under which it is guaranteed to converge and crucially, under which the resulting equilibrium is unique. The algorithm was applied to a local information model, in which all signals are endogenous economic variables. In this model, monetary shocks are non-neutral. An advantage of the model is that the information structure is disciplined by the real structure of the economy, which can be easily estimated with industry-level data.

Signal Operator Iteration may prove valuable for many applications. Macroeconomic models with information frictions that previously relied on exogenous noise, or that made approximations to the information structure, can now be solved with fully endogenous signals. Such models can be used to answer questions that were impossible when information was exogenous. How can a monetary authority influence expectations by affecting endogenous variables? What is the optimal monetary policy in such an environment? What about fiscal stabilization or financial regulation? A wide range of policies that affects asset prices or other endogenous quantities from which agents might draw information can now be addressed.

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A Linearization Details

The Euler equation is

$$\frac{1}{1-\varphi'(\frac{I_{i,t}}{K_{i,t}})} = E_{i,t}\left[\beta\left(\frac{C_{i,t}}{C_{i,t+1}}\right)^{\gamma}\left(\frac{R_{K,i,t+1}}{P_{t+1}}+\frac{1}{1-\varphi'(\frac{I_{i,t+1}}{K_{i,t+1}})}\left(\varphi'(\frac{I_{i,t+1}}{K_{i,t+1}})\frac{I_{i,t+1}}{K_{i,t+1}}-\varphi(\frac{I_{i,t+1}}{K_{i,t+1}})+1-\delta\right)\right)\right] \quad (63)$$

Substituting for the rental rate $R_Ki,t+1$ with capital demand (45) gives

$$\frac{1}{1-\varphi'(\frac{I_{i,t}}{K_{i,t}})} = E_{i,t}\left[\beta\left(\frac{C_{i,t}}{C_{i,t+1}}\right)^{\gamma}\left(\frac{P_{i,t+1}\alpha Y_{i,t+1}}{P_{t+1}K_{i,t+1}}+\frac{1}{1-\varphi'(\frac{I_{i,t+1}}{K_{i,t+1}})}\left(\varphi'(\frac{I_{i,t+1}}{K_{i,t+1}})\frac{I_{i,t+1}}{K_{i,t+1}}-\varphi(\frac{I_{i,t+1}}{K_{i,t+1}})+1-\delta\right)\right)\right] \quad (64)$$

and substituting for the price $P_{i,t+1}$ with sectoral demand (43) gives

$$\frac{1}{1-\varphi'(\frac{I_{i,t}}{K_{i,t}})} = E_{i,t}\left[\beta\left(\frac{C_{i,t}}{C_{i,t+1}}\right)^{\gamma}\left(H_{i,t+1}\frac{\alpha Y_{i,t+1}^{1-1/\eta}}{K_{i,t+1}}+\frac{1}{1-\varphi'(\frac{I_{i,t+1}}{K_{i,t+1}})}\left(\varphi'(\frac{I_{i,t+1}}{K_{i,t+1}})\frac{I_{i,t+1}}{K_{i,t+1}}-\varphi(\frac{I_{i,t+1}}{K_{i,t+1}})+1-\delta\right)\right)\right] \quad (65)$$

Combining labor market clearing with the production function gives $Y_{i,t}=A_{i,t}K_{i,t}^{\alpha}$, which implies

$$\frac{1}{1-\varphi'(\frac{I_{i,t}}{K_{i,t}})} = E_{i,t}\left[\beta\left(\frac{C_{i,t}}{C_{i,t+1}}\right)^{\gamma}\left(H_{i,t+1}\alpha A_{i,t+1}^{1-1/\eta}K_{i,t+1}^{\alpha(1-1/\eta)-1}+\frac{1}{1-\varphi'(\frac{I_{i,t+1}}{K_{i,t+1}})}\left(\varphi'(\frac{I_{i,t+1}}{K_{i,t+1}})\frac{I_{i,t+1}}{K_{i,t+1}}-\varphi(\frac{I_{i,t+1}}{K_{i,t+1}})+1-\delta\right)\right)\right] \quad (66)$$

Log-linearize this equation to get

$$\frac{1}{(1-\varphi'(\frac{\bar{I}}{\bar{K}}))^2}\varphi''(\frac{\bar{I}}{\bar{K}})\frac{\bar{I}}{\bar{K}}(i_{i,t}-k_{i,t}) = E_{i,t}\left[\gamma\left(c_{i,t}-c_{i,t+1}\right)+\beta\alpha\frac{\bar{Y}}{\bar{K}}(h_{i,t+1}+(1-\frac{1}{\eta})a_{i,t+1}+(\alpha(1-\frac{1}{\eta})-1)k_{i,t+1})\right]+ E_{i,t}\left[\beta\frac{\varphi'(\frac{\bar{I}}{\bar{K}})\frac{\bar{I}}{\bar{K}}-\varphi(\frac{\bar{I}}{\bar{K}})+1-\delta}{1-\varphi'(\frac{\bar{I}}{\bar{K}})}\left(\frac{\varphi''(\frac{\bar{I}}{\bar{K}})}{1-\varphi'(\frac{\bar{I}}{\bar{K}})}\frac{\bar{I}}{\bar{K}}+\varphi''(\frac{\bar{I}}{\bar{K}})(\frac{\bar{I}}{\bar{K}})^2\right)(i_{i,t+1}-k_{i,t+1})\right] \quad (67)$$

The law of motion for capital implies $\bar{I}/\bar{K} = \delta$, and by assumption $\varphi'(\delta) = \varphi(\delta) = 0$, so the linearized Euler equation simplifies to

$$\bar{\varphi}(i_{i,t} - k_{i,t}) =$$
$$E_{i,t}\left[\gamma\left(c_{i,t} - c_{i,t+1}\right) + \beta\alpha\frac{\bar{Y}}{\bar{K}}(h_{i,t+1} + (1 - \frac{1}{\eta})a_{i,t+1} + (\alpha(1 - \frac{1}{\eta}) - 1)k_{i,t+1}) + \beta(1-\delta)(1+\delta)\bar{\varphi}(i_{i,t+1} - k_{i,t+1})\right] \quad (68)$$

where I have defined the term $\bar{\varphi} \equiv \varphi''(\delta)\delta$.

To get the Euler equation in terms of $i_{i,t}$ and $k_{i,t}$, consumption must be substituted out using the linearized resource constraint

$$\bar{Y}y_{i,t} = \bar{C}c_{i,t} + \bar{I}i_{i,t} \quad (69)$$

and linearized production function

$$\bar{A}\bar{K}^{\alpha}(a_{i,t} + \alpha k_{i,t}) = \bar{Y}y_{i,t} \quad (70)$$

which implies that consumption is given by $c_{i,t} = \frac{\bar{Y}}{\bar{C}}(a_{i,t} + \alpha k_{i,t}) - \frac{\bar{I}}{\bar{C}}i_{i,t}$. This implies that the final linear Euler equation in terms of capital, investment, and signals is:

$$\bar{\varphi}(i_{i,t} - k_{i,t}) = E_{i,t}\left[\gamma\left(\frac{\bar{Y}}{\bar{C}}(a_{i,t} + \alpha k_{i,t}) - \frac{\bar{I}}{\bar{C}}i_{i,t} - \frac{\bar{Y}}{\bar{C}}(a_{i,t+1} + \alpha k_{i,t+1}) + \frac{\bar{I}}{\bar{C}}i_{i,t+1}\right)\right] + ...$$
$$E_{i,t}\left[\beta\alpha\frac{\bar{Y}}{\bar{K}}(h_{i,t+1} + (1 - \frac{1}{\eta})a_{i,t+1} + (\alpha(1 - \frac{1}{\eta}) - 1)k_{i,t+1}) + \beta(1-\delta)(1+\delta)\bar{\varphi}(i_{i,t+1} - k_{i,t+1})\right] \quad (71)$$

To find $\frac{\bar{Y}}{\bar{K}}$, consider the Euler equation in the deterministic steady state:

$$\frac{1}{1 - \varphi'(\frac{\bar{I}}{\bar{K}})} = \beta\left(\alpha\frac{\bar{Y}}{\bar{K}} + \frac{1}{1 - \varphi'(\frac{\bar{I}}{\bar{K}})}\left(\varphi'(\frac{\bar{I}}{\bar{K}})\frac{\bar{I}}{\bar{K}} - \varphi(\frac{\bar{I}}{\bar{K}}) + 1 - \delta\right)\right)$$

Then, substituting with the law of motion's steady state $\bar{I} = \delta\bar{K}$ and the assumption $\varphi'(\delta) = \varphi(\delta) = 0$ gives

$$1 = \beta\left(\alpha\frac{\bar{Y}}{\bar{K}} + 1 - \delta\right)$$

which implies a steady state output-to-capital ratio of

$$\frac{\rho + \delta}{\alpha} = \frac{\bar{Y}}{\bar{K}}$$

where ρ is defined by $\beta \equiv 1/(1+\rho)$.